Insurance in Human Capital Models with Limited Enforcement

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Abstract

This paper develops a tractable human capital model with limited enforceability of contracts. The model economy is populated by a large number of long-lived, risk-averse households with homothetic preferences who can invest in risk-free physical capital and risky human capital. Households have access to a complete set of credit and insurance contracts, but their ability to use the available financial instruments is limited by the possibility of default (limited contract enforcement). We provide a convenient equilibrium characterization that facilitates the computation of recursive equilibria substantially. We use a calibrated version of the model with three age groups (young, middle-aged, old) that differ in their expected human capital returns. According to the baseline calibration, for young households (age 21-40) only about one half of human capital risk is insured and the welfare loss due to the lack of insurance is 3.5 percent of lifetime consumption. Realistic variations in the model parameters have non-negligible effects on equilibrium insurance and welfare, but the result that young households are severly underinsured is robust to such variations.

Keywords: Human Capital Risk, Limited Enforcement, Insurance

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I. Introduction

Many households own almost nothing but their human capital. Moreover, there is strong evidence that human capital investment is risky, but consumption insurance against this risk is far from complete. More precisely, a significant fraction of labor income is the return to human capital investment, and a voluminous empirical literature has shown that individual households face large and highly persistent labor income shocks that have strong effects on individual consumption. In this paper, we argue that one financial friction, limited contract enforcement, has the potential to explain a substantial part of the observed lack of consumption insurance as an endogenous outcome. Intuitively, in equilibrium households with high human capital returns and little financial wealth would like to borrow in order to buy insurance and invest in human capital, but they cannot do so because of borrowing constraints that arise endogenously due to the limited enforceability of credit contracts.

Our analysis proceeds in two steps. In a first step, we develop a tractable human capital model with limited contract enforcement and provide a useful equilibrium characterization result. In a second step, we use a calibrated version of the model to address the following question: Can a reasonably calibrated macro model with physical capital, human capital and limited contract enforcement explain a substantial fraction of the observed lack of consumption insurance? In this paper, we show that the answer to this question is "yes".

The model developed in this paper is a version of the type of human capital model that has been popular in the endogenous growth literature. More specifically, we consider a production economy with an aggregate constant-returns-to-scale production function using physical and human capital as input factors. There are a large number (a continuum) of individual households with CRRA-preferences who can invest in risk-free physical capital and risky human capital. Human capital investment is risky due to shocks to the stock of human capital that follow a stationary Markov process with finite support (Markov chain). In the main part of the paper, we assume that all shocks are idiosyncratic, but we also discuss how our theoretical characterization result can be extended to the case in which idiosyncratic and aggregate shocks co-exists. Households have access to a complete set of credit and insurance contracts, but their ability to use the available financial instruments is limited by the possibility of default (endogenous borrowing/short-sale constraints). Defaulting households continue to participate in the labor market (non-pledgeability of human capital), but are excluded from financial markets until a stochastically determined future date.

Our general characterization result for recursive equilibria is based on the property that individual consumption policy functions are linear in total wealth (financial plus human) and individual portfolio choices are independent of wealth. Using this linearity result, we show that recursive equilibria can be found by solving a fixed-point problem that is independent of the wealth distribution. Moreover, the maximization problem of individual households is shown to be convex so that a simple FOC-approach is applicable. In short, a rather complex, infinite-dimensional fixed-point problem has been transformed into a relatively simple, finite-dimensional fixed-point problem. Indeed, in many applications, including the one we present in this paper, the equilibrium problem is reduced to a low-dimensional fixed-point problem that can be solved numerically at very low computational cost.

In the quantitative part of the paper, we consider a version of the model with i.i.d. human capital shocks and three age-groups: young households (age 21-40), middle-aged households (age 41-60), and old households (age 61-80). Household age affects expected human capital returns. We identify these returns by requiring the model to match the empirical life-cycle profile of earnings, and the implied human capital returns are in line with the estimates of the empirical literature. The calibrated model is also consistent with the empirical evidence on human capital risk. Specifically, in our model, i.i.d. shocks to the stock of human capital translate into a labor income process that follows a logarithmic random walk, that is, labor income shocks are permanent. The random-walk specification has often been used in the empirical literature to model the permanent component of labor income risk, and we use the estimates obtained by this literature to calibrate our model economy. For the baseline calibration, we use a degree of relative risk aversion of 1 (log-utility) and a level of contract enforcement (exclusion from financial markets in case of default) in line with the US bankruptcy code. The results of our quantitative analysis can be summarized as follows.

First, according to our baseline calibration, young households (age group 21-40) are substantially underinsured. Specifically, young households cannot borrow freely and only insure 43 percent of human capital risk even though insurance markets exist and are perfectly competitive. Further, the welfare consequences of the lack of consumption insurance are severe: the welfare of young households would increase by 3.5 percent of lifetime consumption if they had unlimited access to financial markets. Second, even though realistic variations in the model parameters have non-negligible effects on equilibrium insurance and welfare, the main result that young households are severely underinsured is robust such variations.

In sum, this paper makes a methodological contribution and a substantive contributions. Theoretically, we develop a general framework and prove a characterization result for recursive equilibria that provides a powerful tool for the quantitative analysis of a wide range of interesting macroeconomic issues. Economically, we show that, contrary to the results obtained by most of the previous literature, limited contract enforcement can explain the observed lack of consumption insurance for a large group of households.

Literature This paper builds on the large literature on limited commitment/enforcement. See, for example, Alvarez and Jermann (2000), Kehoe and Levine (1993), Kocherlakota (1996), and Thomas and Worrall (1988) for seminal theoretical contributions and Krueger and Perri (2006) and Ligon, Thomas, and Worrall (2002) for highly influential quantitative work. Our theoretical contribution is to develop a tractable model with human capital accumulation and to show how to avoid the non-convexity problem that often arises in limited enforcement models with production.[1] Our substantive contribution is to show that a calibrated macro model with capital and limited contract enforcement generates a substantial degree of underinsurance. In contrast, the previous literature did not consider life-cycle variations in earnings and human capital investment decisions, and found that consumption insurance is almost perfect in calibrated models with physical capital (Cordoba, 2004, and Krueger and Perri, 2006).[2] In the models used by the previous li models, there is little reason for households to borrow so that enforcement constraints do not bind often and consumption insurance is high. Finally, we share with Andolfatto and Gervais (2006) and Lochner and Monge (2011) the focus on human capital accumulation and endogenous borrowing constraints due to enforcement problems, but we go beyond their work by studying the interaction between borrowing constraints and insurance.

The current paper is most closely related to Krebs, Kuhn, and Wright (2015), who provide evidence from the life insurance market that human capital returns and insurance are negatively correlated. Krebs, Kuhn, and Wright (2015) also conduct a quantitative analysis of under-insurance based on a calibrated macro model similar to the one studied here. The current analysis goes beyond Krebs, Kuhn, and Wright (2015) in two important dimensions. First, the theoretical results derived in the current paper covers the case of general CRRA-preferences, non-steady state behavior, and aggregate risk. In contrast, Krebs, Kuhn, and Wright (2015) confine attention to steady state equilibria in economies with log-preferences and no aggregate risk. Second, in the current paper we provide a comprehensive study of the conditions that generate non-negligible under-insurance in calibrated models with limited

[1] Wright (2001) has shown how to circumvent the non-convexity issue in linear production models (AK-model) with limited enforcement. The model structure we use in this paper is based on the human capital model with incomplete markets analyzed in Krebs (2003).

[2] Krueger and Perri (2006) match the cross-sectional distribution of consumption fairly well, but the implied volatility of individual consumption is negligible in their model.

enforcement and risky human capital investment.

Our paper is also related to the voluminous literature on macroeconomic models with exogenously incomplete markets, and in particular studies of human capital accumulation (Krebs, 2003, Guvenen, Kuruscu, and Ozkan, 2011, and Huggett, Ventura, and Yaron, 2011). The current paper and Krebs, Kuhn, and Wright (2015) are complementary to the incomplete-market literature on human capital investment in the sense that they address similar issues from different angles. Specifically, the incomplete-market approach studies the effect of human capital risk on investment/saving and consumption behavior when no insurance beyond self-insurance is available. In contrast, the limited-enforcement approach analyzes the effect of human capital risk on investment/saving and consumption behavior when insurance markets are available, but endogenous borrowing constraints due to limited contract enforcement generate under-insurance.

II. Model

In this section, we develop the model and define the relevant equilibrium concept.

a) Goods Production and Firms

Time is discrete and indexed by $t = 0, 1, \ldots$. There is one all-purpose good that can be consumed or used for investment purposes. Production of this good is undertaken by a representative firm that rents capital and labor in competitive markets and uses these input factors to produce output, Y_t, according to the aggregate production function $Y_t = F(K_t, H_t)$. Here K_t is the aggregate stock of physical capital and H_t is the aggregate level of efficiency-weighted human capital employed by the firm.

The aggregate production function, F, is a standard neoclassical production function, that is, it has constant-returns-to-scale, satisfies a Inada condition, and is continuous, concave, and strictly increasing in each argument. Given these assumptions on F, the implied intensive-form production function, $f(\tilde{K}) = F(\tilde{K}, 1)$, is continuous, strictly increasing, strictly concave, and satisfies a corresponding Inada condition, where we introduced the "capital-to-labor ratio" $\tilde{K} = K/H$. Given the assumption of perfectly competitive labor and capital markets, profit maximization implies

$$\begin{aligned} \tilde{r}_{kt} &= f'(\tilde{K}_t) \\ \tilde{r}_{ht} &= f(\tilde{K}_t) + f'(\tilde{K}_t)\tilde{K}_t \ , \end{aligned} \tag{1}$$

where $\tilde{r}_k$ is the rental rate of physical capital and $\tilde{r}_h$ is the rental rate of human capital. Note that $\tilde{r}_h$ is simply the wage rate per unit of human capital. Clearly, (1) defines rental rates as functions of the capital to labor ratio: $\tilde{r}_k = \tilde{r}_k(\tilde{K})$ and $\tilde{r}_h = \tilde{r}_h(\tilde{K})$.

b) Capital Accumulation and Aggregate Resource Constraint

One unit of the all-purpose good can be transformed into one unit of physical capital. The accumulation equation for the aggregate stock of physical capital is

$$K_{t+1} = (1 - \delta_k)K_t + X_{kt} \ , \tag{2}$$

where δ_k is the depreciation rate of physical capital and X_{kt} is investment in physical capital.

Production of human capital is undertaken at the household level. There is a continuum of infinitely-lived households of unit mass and each household can transform one unit of the all-purpose good into ϕ units of human capital. The accumulation equation for human capital, h, of an individual household is given by

$$h_{t+1} = (1 + \epsilon(s_t))\, h_t + \phi x_{ht} \ , \tag{3}$$

where x_{ht} is human capital investment of the individual household in period t and s_t is the exogenous state of the individual household in period t. We assume that $\{s_t\}$ follows a Markov shock process and that s_t takes on a finite number of possible values.

We denote by $s^t = (s_1, \dots, s_t)$ the history of individual states s_t up to period t and let $\pi(s^t) = \pi(s_t|s_{t-1}) \dots \pi(s_2|s_1)$ stand for the probability that s^t occurs.

In line with Jones and Manuelli (1990) and Rebelo (1991), the human capital accumulation equation (3) focuses on the (direct) resource cost of human capital production. In contrast, Lucas (1988) and more recently Huggett et al. (2011) and Lochner and Monge (2011) assume that the only cost of human capital production is a time cost. As suggested by Ben-Porath (1967) and Trostel (1995), in many applications both resource cost and time cost are important components of the total cost of human capital production. It is straightforward to extend our model to the case that allows for both goods cost and time cost of human capital production.

The term ϵ in (3) captures deterministic and random changes in human capital that are due to depreciation, learning by doing, and various shocks to human capital (skills) of households. For example, a negative human capital shock could can occur when a household member loses firm- or sector-specific human capital subsequent to job termination (worker displacement). A decline in health (disability) or death of a household member provide further examples of negative human capital shocks. In this case, both general and specific human capital are lost. Internal promotions and upward movement in the labor market provide two examples of positive human capital shock.

In the quantitative example considered in Section IV, we assume that the individual state has two components, $s_t = (s_{1t}, s_{2t})$. The first component, s_{1t}, determines household age and the second component, s_{2t}, represents human capital shocks capturing labor market risk. We further assume $\epsilon(s_{1t}, s_{2t}) = \varphi(s_{1t}) - \delta_h + \eta(s_{2t})$. We interpret $\delta_h > 0$ as the general depreciation of human capital and $\varphi(s_1) > 0$ as learning-by-doing that depends on household age s_1.

The aggregate resource constraint states that total output produced is equal to aggregate consumption plus aggregate investment (in physical and human capital):

$$Y_t \;=\; C_t + X_{kt} + X_{ht} \tag{4}$$

where $Y_t = F(K_t, H_t)$, $C_t = E[c_t]$, $X_{ht} = E[x_{ht}]$, and $H_t = E[z_t h_t]$. Here $z_t = z(s_t)$ is variable that measures the productivity/efficiency of the human capital of an individual household in period t.

c) Household Budget Constraint

In each period, households can buy and sell a (sequentially) complete set of financial contracts (assets) with state-contingent payoffs, and we assume that for each state s there is one contract (Arrow security). We denote by $a_{t+1}(s_{t+1})$ the quantity bought in period t (sold if negative) of the contract that pays off one unit of the good in period $t+1$ if s_{t+1} occurs. A budget-feasible plan has to satisfy the sequential budget constraint

$$\begin{aligned}
\tilde{r}_{ht} z(s_t) h_t + a_t(s_t) &= c_t + x_{ht} + \sum_{s_{t+1}} a_{t+1}(s_{t+1}) q_t(s_{t+1}) \\
h_{t+1} &= (1 + \epsilon(s_t))\, h_t + \phi x_{ht} \\
h_{t+1} + \sum_{s_{t+1}} a_{t+1}(s_{t+1}) q_t(s_{t+1}) &\geq 0 \\
c_t \geq 0 \;\; , \;\; h_{t+1} \geq 0 &
\end{aligned} \tag{5}$$

where $q_t(s_{t+1})$ is the price of a financial contract that pays off if s_{t+1} occurs. Note that the first inequality in (5) requires total wealth (human plus financial) to be non-negative. In our setting, this constraint is sufficient to rule out Ponzi-schemes.

An individual household begins life in period $t = 0$ with an initial endowment (a_0, h_0). The initial state of an individual household is a vector (a_0, h_0, s_0). Given the initial state (a_0, h_0, s_0), the household chooses a plan $\{c_t, a_t, h_t\}$, where each plan is a sequence of functions mapping histories, s^t, into actions, $c_t(s^t)$, $a_{t+1}(s^t, .)$, and $h_{t+1}(s^t)$. Note that all household level equations have to hold in realizations, that is, they have to hold for all histories, s^t.

d) Preferences

Households are infinitely-lived and have identical preferences over consumption plans. Households are risk-averse and their preferences allow for a time-additive expected utility representation:[3]

$$U(\{c_t\}|s_0) \doteq E\left[\sum_{t=0}^{\infty}\beta^t u(c_t)|s_0\right] , \tag{6}$$

where the expectations in (6) is taken over all histories, s^t, keeping the initial type, s_0, fixed. We assume that the one-period utility function exhibits constant relative risk aversion: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ and $u(c) = lnc$ otherwise. In other words, we assume that preferences are homothetic.

e) Participation/Enforcement Constraint

We confine attention to equilibria in which households have no incentive to default. Thus, households have to satisfy the sequential enforcement (participation) constraint, that is, for all t and all s^t we have:

$$E\left[\sum_{n=0}^{\infty}\beta^n u(c_{t+n})|s^t\right] \geq V_d(h_t(s^{t-1}), s_t) \tag{7}$$

where V_d is the continuation value of a household who decides to default in period t. This default value is determined as follows.

We assume that upon default all debts of the household are canceled and all financial assets seized so that $a_t(s_t) = 0$. Following default, households are excluded from purchasing insurance contracts and borrowing (going short). We assume that exclusion continues until a stochastically determined future date that occurs with probability $(1-p)$ in each period; that

[3]The notation $E\left[\sum_{t=0}^{\infty}\beta^t u(c_t)|s_0\right]$ stands for $\sum_{s^t}\sum_{t=0}^{\infty}\beta^t u(c_t(s^t))\pi(s^t|s_0)$.

is, the probability of remaining in (financial) autarky is p. Following a default, households retain their human capital and continue to earn the wage rate r_h per unit of human capital. In addition, they can still invest in human capital. After regaining access to financial markets, the households expected continuation value is V^e, which depends on h and s at the time of exiting default ($a = 0$ at that point in time). For the individual household the function V^e is taken as given, but we close the model and determine this function endogenously by requiring that $V^e = V$, where V is the equilibrium value function associated with the maximization problem of a household who participates in financial markets.[4]

In sum, V_d is the value function associated with the following household maximization problem

$$V_d(h_t, s_t) = \max_{\{c_{t+n}, h_{t+n}|h_t, s_t\}} \left\{ E\left[\sum_{n=0}^{\infty} (p\beta)^n u(c_{t+n})|h_t, s_t\right] + \beta(1-p)E\left[\sum_{n=1}^{\infty}(1-p^n)\beta^n V^e_{t+n}(h_{t+n}, s_{t+n})|h_t, s_t\right]\right.$$

where the continuation plan $\{c_{t+n}, h_{t+n}|h_t, s_t\}$ has to satisfy the sequential budget constraint

$$\begin{aligned} \tilde{r}_{h,t+n} z(s_{t+n}) h_{t+n} &= c_{t+n} + x_{h,t+n} \\ h_{t+n+1} &= \left(1 + \epsilon(s_{t+n})\right) h_{t+n} + \phi x_{h,t+n} \\ c_{t+n} \geq 0 \quad , \quad h_{t+n+1} &\geq 0 \end{aligned} \tag{8}$$

f) Household Decision Problem

The initial state of an individual household, which defines the household type, is the vector (a_0, h_0, s_0). For given initial state, a household chooses a plan $\{c_t, a_t, h_t\}$. The set of budget

[4]The previous literature has usually assumed $p = 1$ (permanent autarky). See, however, Krueger and Uhlig (2006) for a model with $p = 0$ following a similar approach to ours. Note also that the credit (default) history of an individual household is not a state variable affecting the expected value function, V^e; we assume that the credit (default) history of households is information that cannot be used for contracting purposes. This is in keeping with the U.S. bankruptcy code which limits the history of past behavior that can be retained in credit reports.

feasible household plans is defined as

$$B(a_0, h_0, s_0) \doteq \{ \{c_t, a_t, h_t\} \,|\, \{c_t, a_t, h_t\} \; satisfies \; (3), (5), \; and \; (7) \} \tag{9}$$

The decision problem of a household of type (a_0, h_0, s_0) is

$$\begin{aligned} &\max_{\{c_t, a_t, h_t\}} U\left(\{c_t\}|s_0\right) \\ s.t. \;\; &\{c_t, a_t, h_t\} \in B(a_0, h_0, s_0) \end{aligned} \tag{10}$$

g) Equilibrium

We confine attention to equilibria in which financial contracts are priced in a risk-neutral manner,

$$q_t(s_{t+1}) = \frac{\pi(s_{t+1}|s_t)}{1 + r_{ft}} \, , \tag{11}$$

where r_f is the interest rate on financial transactions, which is equal to the return on physical capital investment, $r_{ft} = \tilde{r}_{kt} - \delta_k$. The pricing equation (11) can be interpreted as a zero-profit condition. More precisely, consider financial intermediaries that sell insurance contracts to individual households and invest the proceeds in the risk-free asset that can be created from the complete set of financial contracts and yields a certain return r_f. Given that financial intermediaries face linear investment opportunities and assuming no quantity restrictions on the trading of financial contracts for financial intermediaries, equilibrium requires that financial intermediaries make zero profit, namely condition (11).

Market clearing requires that the aggregate stock of physical capital employed by the representative firm is equal to the value of financial wealth held by households. More precisely, the capital market clearing condition reads:

$$K_{t+1} = E[\sum_{s_{t+1}} a_{t+1}(s_{t+1}) q_t(s_{t+1})] \tag{12}$$

It is straightforward to show that the capital market clearing condition (12) in conjunction with the household budget constraint (5) and the capital accumulation equations (2) and (3) imply the goods market clearing condition (4) using the asset pricing formula (11).

We take an initial distribution over households types, (a_0, h_0, s_0) as given. We place no restrictions on this initial distribution. Our definition of a sequential equilibrium is standard:

Definition 1 A sequential equilibrium is a sequence of aggregate physical capital, $\{K_t\}$, and rental rates, $\{\tilde{r}_{kt}, \tilde{r}_{ht}\}$, and a family of household plans, $\{c_t, a_t, h_t, d_t\}$, one for each initial type (a_0, h_0, s_0), so that

i) Utility maximization of households: for each initial state, (a_0, h_0, s_0), the plan $\{c_t, a_t, h_t\}$ solves the household problem (10).

ii) Profit maximization of firms: the aggregate capital-to-labor ratio and rental rates satisfy the first-order conditions (1) for all t

iii) Profit maximization of financial intermediaries: financial contracts are priced according to (11)

iv) Market clearing: aggregate physical capital used by the representative firm is equal to aggregate financial wealth owned by households – equation (12) holds.

v) Rational expectations: expected continuation value functions are equal to actual continuation value functions: $V_t^e = V_t$.

We next turn to the characterization of equilibria.

III. Theoretical Results

In this section, we state the main theoretical results.

a) Change of Variables

For the characterization of equilibria, it is convenient to introduce new variables that emphasize that individual households solve a standard inter-temporal portfolio choice problem (with additional participation constraints). To this end, introduce the following variables:

$$
\begin{aligned}
w_t &= h_t + \sum_{s_t} q_{t-1}(s_t) a_t(s_t) \\
\theta_{ht} &= \frac{h_t}{\phi w_t} \;\;,\;\; \theta_{at}(s_t) = \frac{a_t(s_t)}{w_t} \\
1 + r_t &= (1 + r_{ht}(s_t))\theta_{ht} + \theta_{at}(s_t)
\end{aligned}
\tag{13}
$$

where $r_{ht}(s_t) \doteq z(s_t)\phi\tilde{r}_{ht} + \epsilon(s_t)$ is the return on human capital investment. In (13) the variable w_t stands for beginning-of-period wealth consisting of human wealth, h_t, and financial wealth, $\sum_{s_t,S_t} q_{t-1}(s_t)a_t(s_t)$. The variable $\theta_t = (\theta_{ht}, \theta_{at})$ denotes the vector of portfolio shares and $(1 + r)$ is the total return to investment. Using the new notation and substituting out the investment variables, x_{kt} and x_{ht}, the budget constraint (5) reads

$$
\begin{aligned}
w_{t+1} &= (1 + r(\theta_t, s_t, d_t)\, w_t - c_t \\
1 &= \theta_{h,t+1} + \sum_{s_{t+1}} q_t(s_{t+1})\theta_{a,t+1}(s_{t+1}) \\
c_t &\geq 0 \;\;,\;\; w_{t+1} \geq 0 \;\;,\;\; \theta_{h,t+1} \geq 0\,.
\end{aligned}
\tag{14}
$$

Clearly, (14) is the budget constraint corresponding to an inter temporal portfolio choice problem with linear investment opportunities and no exogenous source of income.

Using the definition (13), simple algebra shows that the household maximization problem (10) is equivalent to the household maximization problem

$$
\begin{aligned}
&\max_{\{c_t, w_t, \theta_t\}} U(\{c_t\}|s_0) \\
&s.t.\;\; \{c_t, w_t, \theta_t\} \in B(w_0, \theta_0, s_0)
\end{aligned}
\tag{15}
$$

where the budget set is now defined as

$$
B(w_0, \theta_0, s_0) \doteq \{\{c_t, w_t, \theta_t\} \,|\, \{c_t, w_t, \theta_t\} \; satisfies \; (7) \; and \; (14)\}
$$

b) Recursive Equilibrium: Definition

We next define a recursive equilibrium. The household maximization problem (14) suggests that we can use (w, θ, s) as the individual state variable. For the aggregate state, the variable $\Omega \in \mathbb{R}^n$ defined as

$$\Omega_t(s_t) \doteq \frac{E\left[(1+r_t)w_t|s_t\right]}{E\left[(1+r_t)w_t\right]} .$$

turns out to be a sufficient variable (see below), where $\Omega_t(s_t)$ is the share of aggregate total wealth owned by all households of type s_t. Note that $E[\Omega_t] = \sum_{s_t} \Omega_t(s_t) = 1$ and that $(1+r_t)w_t$ is individual wealth including current asset payoffs. In the recursive equilibrium we construct below, the evolution of the endogenous aggregate state variable is given by an endogenous law of motion $\Omega_{t+1} = \Phi(\Omega_t)$. We further show that next-period's optimal portfolio choice is independent of (w_t, θ_t), which implies that the market clearing conditions (12) can be re-written as

$$\tilde{K}_{t+1} = \frac{\sum_{s_t}\left(1-\theta_h(s_t,\Omega_t)\right)\Omega_t(s_t)}{\sum_{s_t}\theta_h(s_t,\Omega_t)\Omega_t(s_t)} \tag{16}$$

Equation (16) defines a function $\tilde{K} = \tilde{K}(\Omega)$, which together with the first-order conditions (1) defines rental rate functions $\tilde{r}_k = \tilde{r}_k(\Omega)$ and $\tilde{r}_h = \tilde{r}_h(\tilde{K})$.

Given our definition of sequential equilibrium and the variables defined so far, our definition of recursive equilibrium is standard:

Definition 2 A recursive equilibrium is a law of motion, Φ, for the aggregate state variable, Ω, rental rate functions $\tilde{r}_k = \tilde{r}_k(\Omega)$ and $\tilde{r}_h(\Omega)$, an expected value function, V^e, and a household policy function, g,[5] so that

i) Utility maximization of households: for each household type, $(w_0, \theta_0, s_0) = (w_0, \theta_0, s_0)$, the household policy function, g, generates a plan, $\{c_t, w_t, \theta_t\}$, that solves the household

[5]The function g defines next period's endogenous state as a function of this period's endogenous state and this period's exogenous shock:$(w_{t+1}, \theta_{t+1}) = g(w_t, \theta_t, s_t)$.

maximization problem (14).

ii) Profit maximization of firms: aggregate capital-to-labor ratio and rental rates satisfy the first-order conditions (1).

iii) Financial intermediation: financial contracts are priced according to (11)

iv) Market clearing: equation (16) holds.

v) Rational expectations: $V^e = V$ and Φ is the law of motion induced by g.

c) Characterization of Household Problem (Partial Equilibrium)

The principle of optimality in conjunction with our discussion in the previous section regarding the appropriate aggregate state suggest that the household maximization problem (14) is equivalent to the Bellman equation

$$
\begin{aligned}
V(w,\theta,s,\Omega) \quad &= \quad \max_{w',\theta'} \left\{ u\left((1+r(\theta,s,\Omega))w - w'\right) + \beta \sum_{s'} V\left(w',\theta',s',\Omega'\right)\pi(s'|s) \right\} \\
s.t. \quad 1 \ &= \ \theta'_h + \sum_{s'} \frac{\pi(s'|s)\theta'_a(s')}{1+r_f(\Omega)} \\
0 \ &\le \ w' \ \le \ (1+r(\theta,s,\Omega))w \ , \ \theta'_h \ge 0 \\
V(w',\theta',s',\Omega') \ &\ge \ V_d(w',\theta'_h,s',\Omega') \\
\Omega' \ &= \ \Phi(\Omega)
\end{aligned} \tag{17}
$$

where the default value function is given by

$$
\begin{aligned}
V_d(w,\theta_h,s,\Omega) \quad &= \quad \max_{w'} \Big\{ u\left((1+r_h(s,\Omega))\theta_h w - w'\right) + \beta p \sum_{s'} V_d\left(w',1,s',\Omega'\right)\pi(s'|s) \\
&\quad + \beta(1-p)\sum_{s'} V^e\left(w',s',\Omega'\right)\pi(s'|s) \Big\} \\
0 \ &\le \ w' \ \le \ (1+r_h(s,\Omega))\theta_h w \\
\Omega' \ &= \ \Phi(\Omega)
\end{aligned}
$$

Let T be the operator associated with the Bellman equation (17). In contrast to the standard case without a participation constraint, the Bellman operator, T, defined by equation (17) is in general not a contraction. However, it is still a monotone operator. Monotone operators might have multiple fixed points, but under certain conditions we can construct a sequence that converges to the maximal element of the set of fixed points. This maximal solution is also the value function (principle of optimality). More precisely, if the condition that for all s

$$\begin{array}{lllll} \forall\theta' : & \beta \sum_{s'} (1 + r(\theta', s', \Omega'))^{1-\gamma} \pi(s'|s) & < & 1 & if \;\; 0 < \gamma < 1 \\ \exists\theta' : & \beta \sum_{s'} (1 + r(\theta', s', \Omega'))^{1-\gamma} \pi(s'|s) & < & 1 & if \;\; \gamma > 1 \end{array} \tag{18}$$

holds,[6] then we have the following results:

Proposition 1. Suppose that condition (18) is satisfied and that the law of motion, Φ, and the value function of a household in financial autarky, V_d, are continuous. Let T stand for the operator associated with the Bellman equation (17). Then

i) There is a unique continuous solution, V_0, to the Bellman equation (18) without participation constraint.

ii) $\lim_{n\to\infty} T^n V_0 = V_\infty$ exists and is the maximal solution to the Bellman equation (17)

iii) V_∞ is the value function, V, of the sequential household maximization problem.

Proof. See appendix.

Consider the case $V^e = V$. Using proposition 2 and an induction argument, we can then show that the value function, V, has the functional form

$$V(w, \theta, s, \Omega) = \begin{cases} \tilde{V}(s, \Omega)(1 + r(\theta, s, \Omega))^{1-\gamma} w^{1-\gamma} & if \;\gamma \neq 1 \\ \tilde{V}(s, \Omega) + \frac{1}{1-\beta} log\,(1 + r(\theta, s, \Omega)) + \frac{1}{1-\beta} w & otherwise \end{cases} \tag{19}$$

[6]Note that for the log-utility case, no condition of the type (18) is required.

and that the corresponding optimal policy function, g, is

$$c(w,\theta,s) \;=\; \begin{cases} \tilde{c}(s,\Omega)(1+r(\theta,s,\Omega))w & if\ \gamma \neq 1 \\ (1-\beta)(1+r(\theta,s,\Omega))w & otherwise \end{cases} \tag{20}$$

$$w'(w,\theta,s,\Omega) \;=\; \begin{cases} (1-\tilde{c}(s,\Omega)(1+r(\theta,s,\Omega))w & if\ \gamma \neq 1 \\ \beta(1+r(\theta,s,\Omega))w & otherwise \end{cases}$$

$$\theta'(w,\theta,s,\Omega) \;=\; \theta'(s,\Omega)\ .$$

In other words, the value function has the functional form of the underlying utility function, consumption and next-period wealth are linear functions of this-period wealth, and next-period portfolio choices only depend on the current shock. Moreover, we also that the intensive-form value function, $\tilde{V}$, together with the optimal consumption and portfolio choices, $\tilde{c}$ and θ, can be found by solving an intensive-form Bellman equation that reads

$$\begin{aligned} \tilde{V}(s,\Omega) \;&=\; \max_{\tilde{c},\theta'} \left\{ \frac{\tilde{c}^{1-\gamma}}{1-\gamma} + \beta(1-\tilde{c})^{1-\gamma} \sum_{s'} \left(1+r(\theta',s',\Omega)\right)^{1-\gamma} \tilde{V}(s',\Omega')\pi(s'|s) \right\} \\ s.t.\quad 1 \;&=\; \theta'_k + \theta'_h + \sum_{s'} \frac{\theta'_a(s')\pi(s'|s)}{1+r_f(\Omega)} \ ,\quad 0 \;\leq\; \tilde{c} \;\leq\; 1 \ ,\quad \theta'_h \geq 0 \\ &\left(\frac{\tilde{V}(s',\Omega')}{\tilde{V}_d(s',\Omega')}\right)^{\frac{1}{1-\gamma}} (1+r(\theta',s',\Omega)) \;\geq\; (1+r_h(s',\Omega))\theta'_h \\ &\Omega' \;=\; \Phi(\Omega) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \tilde{V}_d(s,\Omega) \;=\; \max_{\tilde{c}_d} \Bigg\{ & \frac{\tilde{c}_d^{1-\gamma}}{1-\gamma} + p\beta(1-\tilde{c}_d)^{1-\gamma} \sum_{s'} \left(1+r_h(s',\Omega)\right)^{1-\gamma} \tilde{V}_d(s',\Omega')\pi(s'|s) \\ & (1-p)\beta(1-\tilde{c}_d)^{1-\gamma} \sum_{s'} \left(1+r_h(s',\Omega)\right)^{1-\gamma} \tilde{V}(s',\Omega')\pi(s'|s) \Bigg\} \end{aligned}$$

for $\gamma \neq 1$. In the case of log-utility, the intensive-form Bellman equation reads

$$\begin{aligned}
\tilde{V}(s,\Omega) &= \max_{\theta'} \Big\{ log(1-\beta) + \frac{\beta}{1-\beta} log\beta + \frac{\beta}{1-\beta} \sum_{s'} log(1 + r(\theta', s', \Omega)\pi(s'|s) \\
&\quad + \beta \sum_{s'} \tilde{V}(s', \Omega')\pi(s'|s) \Big\}
\end{aligned}$$

$$s.t. \quad 1 = \theta'_h + \sum_{s'} \frac{\theta'_a(s')\pi(s'|s)}{1 + r_f(\Omega)} \;\; , \quad \theta'_h \geq 0$$

$$e^{(1-\beta)\left(\tilde{V}(s',\Omega') - \tilde{V}_d(s',\Omega')\right)} (1 + r(\theta', s', \Omega) \;\geq\; (1 + r_h(s', \Omega))\theta'_h$$

$$\Omega' = \Phi(\Omega)$$

and

$$\begin{aligned}
\tilde{V}_d(s,\Omega) &= log(1-\beta) + \frac{\beta}{1-\beta} log\beta + \frac{\beta}{1-\beta} \sum_{s'} log(1 + r_d(s', \Omega)\pi(s'|s) \\
&\quad + p\,\beta \sum_{s'} \tilde{V}_d(s', \Omega')\pi(s'|s) + (1-p)\beta \sum_{s'} \tilde{V}(s', \Omega')\pi(s'|s)
\end{aligned}$$

Proposition 2. Suppose that condition (18) is satisfied, the law of motion, Φ, is continuous, and $V^e = V$. Then the value function, V, has the functional form (19) and the optimal policy function is given by (20). Moreover, the intensive-form value function, $\tilde{V}$, and the corresponding optimal consumption and portfolio choices, $\tilde{c}$ and θ', are the maximal solution to the intensive-form Bellman equation (21). This maximal solution is obtained by iterating on the solution, $\tilde{V}_0$, of the intensive-form Bellman equation (21) without participation constraint:

$$\tilde{V} = \lim_{n \to \infty} \tilde{T}^n \tilde{V}_0 \; ,$$

where $\tilde{T}$ is the operator associated with the intensive-form Bellman equation (21)

Proof Appendix.

Note that proposition 2 cannot simply be proved by the guess-and-verify method we have used to prove proposition 1 for the case of financial autarky. The reason is that there may be multiple solutions to the Bellman equation (17). In other words, the operator associated with the Bellman equation is monotone, but not a contraction. However, proposition 2 ensures that we have indeed found the value function associated with the original utility maximization problem, and also provides us with a iterative method to compute this solution. Note further that the constraint set in (21) is linear since the return functions are linear in θ. Thus, the constraint set is convex and we have transformed the original utility maximization problem into a convex problem. In other words, the non-convexity problem alluded to in the introduction has been resolved.

d) Characterization of Recursive Equilibria

Proposition 2 shows how to rewrite the maximization problem of individual households into a recursive problem that is wealth-independent. One implication of the intensive-form representation of the individual maximization problem is that optimal portfolio choices can be written as a function of s only: $\theta' = \theta'(s)$. This result in turn implies that the market clearing condition (12) can be written as (14) defining a function $\tilde{K}' = \tilde{K}'(\Omega)$ and, using the first-order conditions (1), rental rate functions $\tilde{r}'_k = \tilde{r}'_k(\Omega)$ and $\tilde{r}'_h = \tilde{r}'_h(\Omega)$. A second implication of proposition 2 is that the equilibrium law of motion, Φ, can be explicitly derived:

$$\Omega'(s') = \frac{\sum_s (1 - \tilde{c}(s,\Omega))(1 + r(\theta'(s,\Omega), s', \Omega))\pi(s'|s)\Omega(s)}{\sum_{s,s'} (1 - \tilde{c}(s,\Omega))(1 + r(\theta'(s,\Omega), s', \Omega))\pi(s'|s)\Omega(s)} \tag{22}$$

In sum, a recursive equilibrium can be found by solving (21)-(24):

Proposition 3. Suppose that $(\theta, \tilde{c}, \tilde{V}, r_f)$ is an intensive-form equilibrium, that is, the consumption-portfolio choice $(\tilde{c}, \theta)$ together with the intensive-form value function $\tilde{V}$ are the maximal solution to the intensive-form Bellman equation (21), the market clearing conditions (16) is satisfied, and the law of motion is given by (22). Then $(g, \tilde{V}, r_f, \Phi)$ is a recursive equilibrium, where g is the individual policy function associated with $(\theta, \tilde{c})$ and Φ the aggregate law of motion induced by $(\theta, \tilde{c})$.

Proof. See appendix.

Proposition 3 simplifies the computation of recursive equilibria substantially. In our framework, the infinite-dimensional wealth distribution is not a revenant state variable. Instead, the distribution of wealth shares over household types, Ω, becomes a relevant state variable. Note that Ω is in many applications a low-dimensional object. For example, suppose that $s_t = (s_{1t}, s_{2t})$, where $\{s_{1t}\}$ and $\{s_{2t}\}$ are two independent processes and $\{s_{2t}\}$ is an i.i.d process. In this case neither $\tilde{c}$ nor θ depend on s_2 and the relevant aggregate state is $\Omega(s_1)$ only.

e) Extension: Aggregate Shocks

So far , we have considered economies with only idiosyncratic risk, but it is straightforward to introduce aggregate risk into the framework. To this end, suppose that there are idiosyncratic shocks, s, and aggregate shocks, S, and that uncertainty is described by a stationary joint Markov process $\{s_t, S_t\}$ with transition probabilities denoted by $\pi(s_{t+1}, S_{t+1}|s_t, S_t)$. The relevant aggregate state then becomes (Ω_t, S_t), where Ω_t is defined as before. In a recursive equilibrium, the evolution of the endogenous aggregate state variable is given by an endogenous law of motion $\Omega_{t+1} = \Phi(\Omega_t, S_t, S_{t+1})$. Further, rentals rates and the interest rate on financial transactions become functions of the aggregate state: $r_{k,t+1} = r_k(\Omega_t, S_t, S_{t+1})$, $r_{h,t+1} = r_h(\Omega_t, S_t, S_{t+1})$, and $r_{ft} = r_f(\Omega_t, S_t)$. The definition of a recursive equilibrium is, *mutatis mutandis*, as before.

A straightforward (though lengthy) extension of the subsequent theoretical analysis shows that a modified version of our general characterization results still hold. In particular, recursive equilibria can be computed by solving a convex problem that is independent of the wealth distribution, though clearly the finite-dimensional distribution of relative wealth, Ω, still enters into the equilibrium conditions.

IV. Quantitative Analysis

In this section, we present the quantitative example.

a) Specification

We set the period length to one year. We assume that the exogenous individual state has two components, $s = (s_1, s_2)$. The first component, s_1, is the age of a household, which can take on three values: $s_1 \in \{y, m, o\}$. Here $s_1 = o$ stands for a young household, $s_1 = m$ for a middle-aged household, and $s_1 = o$ for an old household. We assume that households age stochastically, that is, transitions from one age group to another age group are governed by transition probabilities $\pi(s_1'|s_1)$. In our calibration below we choose the transition across age groups so that households spend on average 20 years in each age group corresponding to the age groups $20-40$, $41-60$, and $61-80$. Thus, we assume $\pi(y|y) = \pi(m|m) = \pi(o|o) = 1/20$. We interpret the transition from $s_1 = o$ to $s_1 = y$ as the event that an old household consisting of grandparents dies and simultaneously the grandchildren enter the labor market and create a new, young, household. If grandparents care about their grandchildren (intergenerational link in preferences), then this is equivalent to an old household becoming a young household.

The second component of the state, s_2, describes a human capital shock. We assume that human capital shocks are i.i.d. over time and across household types, s_2. We parameterize the ϵ-function appearing in the human capital equation (3) as $\epsilon(s_1, s_2) = \varphi(s_1) - \delta_h + \eta(s_2)$. Note that we assume that the function η is independent of age s_1, an assumption that we can easily relax. We interpret φ as a learning-by-doing parameter and δ_h as the average depreciation rate of human capital in the economy. Note that learning-by-doing depends on age.[7] In our calibration below, we assume that learning-by-doing is more important for young households: $\varphi_y > \varphi_m > \varphi_o$. The human capital return is given by $r_h(s_1, s_2) = \phi\tilde{r}_h + \varphi(s_1) - \delta_h + \eta(s_2)$. Finally, we normalize the mean of human capital shocks to zero: $\sum_{s_2} \eta(s_2)\pi(s_2) = 0$. Thus, expected human capital returns for a household of age s_1 are $\bar{r}_h(s_1) = \sum_{s_2} r_h(s_1, s_2)\pi(s_2) = \tilde{r} + \varphi(s_1) - \delta_h$.

b) Equilibrium Conditions

Given this assumption, the intensive-form Bellman equation (21) becomes

$$\begin{aligned} \tilde{V}(s_1) &= \max_{\tilde{c},\theta'} \left\{ \frac{\tilde{c}^{1-\gamma}}{1-\gamma} + \beta(1-\tilde{c})^{1-\gamma} \sum_{s_1',s_2'} \left(1 + r(\theta', s_1', s_2')\right)^{1-\gamma} \tilde{V}(s_1')\pi(s_2')\pi(s_1'|s_1) \right\} \\ & s.t. \quad 1 \;=\; \theta_h' + \sum_{s_1',s_2'} \frac{\theta_a'(s_1', s_2')\pi(s_2')\pi(s_1'|s_1)}{1 + r_f} \;\;,\;\; 0 \;\le\; \tilde{c} \;\le\; 1 \;\;,\;\; \theta_h' \ge 0 \\ & \left(\frac{\tilde{V}(s_1')}{\tilde{V}_d(s_1')} \right)^{\frac{1}{1-\gamma}} (1 + r(\theta', s_1', s_2')) \;\ge\; (1 + r_h(s_1', s_2'))\theta_h' \end{aligned} \tag{23}$$

and

$$\begin{aligned} \tilde{V}_d(s_1) &= \max_{\tilde{c}_d} \left\{ \frac{\tilde{c}_d^{1-\gamma}}{1-\gamma} + p\beta(1-\tilde{c}_d)^{1-\gamma} \sum_{s_1',s_2'} \left(1 + r_h(s_1', s_2')\right)^{1-\gamma} \tilde{V}_d(s_1')\pi(s_2')\pi(s_1'|s_1) \right. \\ & \left. (1-p)\beta(1-\tilde{c}_d)^{1-\gamma} \sum_{s_1',s_1'} \left(1 + r_h(s_1', s_2')\right)^{1-\gamma} \pi(s_2')\tilde{V}(s_1)\pi(s_1'|s_1) \right\} \end{aligned}$$

From (23) it immediately follows that the optimal portfolio choice, θ, and the optimal consumption-saving choice, $\tilde{c}$, only depend on age s_1 but not on human capital shocks s_2. In other words, household consumption and portfolio choices are independent are independent

[7] Note that we have set the labor productivity parameter $z = 1$ so that all labor income risk is generated through the human capital shock η. In Krebs, Kuhn, and Wright (2015) we consider the more general version with additional shocks z.

of i.i.d. shocks. This then implies that the relevant aggregate state, Ω, only depends on age, s_1. The stationary Ω is then determined by the equation

$$\Omega(s_1') \;=\; \frac{\sum_{s_1,s_2'}(1-\tilde{c}(s_1))(1+r(\theta'(s_1),s_1',s_2'))\pi(s_2')\pi(s_1'|s_1)\Omega(s_1)}{\sum_{s_1,s_1',s_2'}(1-\tilde{c}(s_1))(1+r(\theta'(s_1),s_1',s_2'))\pi(s_2')\pi(s_1'|s_1)\Omega(s_1)}\;. \tag{24}$$

The market clearing conditions (16) becomes

$$\tilde{K} \;=\; \frac{1-\sum_{s_1}\theta_h(s_1)\Omega(s_1)}{\sum_{s_1}\theta_h(s_1)\Omega(s_1)} \tag{25}$$

Equations (23), (24), and (25) determine an intensive-form equilibrium.

c) Computation

We solve the three equations (23), (24), and (25) as follows. We pick an aggregate capital-to-labor ratio, $\tilde{K}$, which determines the rental rates $\tilde{r}_k$ and $\tilde{r}_h$ and therefore also the investment return function r. Given the values for the investment returns, we solve the intensive-form household decision problem (23) and the stationary law of motion (24). This determines values for θ, $\tilde{c}$, and Ω, which can be used to determine a new value for $\tilde{K}$ using (25). We then iterate until convergence.

Our computation approach requires in each step of the iteration to solve the constrained maximization problem (23). We solve this problem by iteration. More precisely, we define the values $\tilde{v}^n(s_1)$ and $\tilde{v}_d^n(s_1)$, recursively by

$$\begin{aligned}\tilde{v}^{n+1}(s_1) \;&=\; \Bigg\{\frac{(\tilde{c}^n(s_1))^{1-\gamma}}{1-\gamma}+ \\ &\qquad \beta\,(1-\tilde{c}^n(s_1))^{1-\gamma}\sum_{s_1',s_2'}\left(\theta_h^n(s_1)(1+r_h(s_1',s_2'))+\theta_a^n(s_1,s_1',s_2')\right)^{1-\gamma}\tilde{v}^n(s_1)\pi(s_2')\pi(s_1'|s_1)\Bigg\}\end{aligned} \tag{26}$$

$$\tilde{c}^n(s_1) \;=\; 1-\left(\beta\sum_{s_1',s_2'}\left(\theta_h^n(s_1)(1+r_h(s_1',s_2'))+\theta_a^n(s_1,s_1',s_2')\right)^{1-\gamma}\pi(s_2')\pi(s_1'|s_1)\right)^{\frac{1}{\gamma}}$$

and

$$\tilde{v}_d^{n+1}(s_1) \;=\; \frac{(\tilde{c}_d(s_1))^{1-\gamma}}{1-\gamma}\;+\;\beta(1-\tilde{c}_d(s_1))^{1-\gamma}p\sum_{s_1',s_2'}\left(1+r_h(s_1',s_2')\right)^{1-\gamma}\tilde{v}_d^n(s_1')\pi(s_2')\pi(s_1'|s_1)$$

$$+\beta(1-\tilde{c}_d(s_1))^{1-\gamma}(1-p)\sum_{s_1',s_2'}\left(1+r_h(s_1',s_2')\right)^{1-\gamma}\pi(s_2')\tilde{v}^n(s_1')\pi(s_1'|s_1)$$

$$\tilde{c}_d(s_1) \;=\; 1-\left(\beta\sum_{s_1',s_2'}\left(1+r_h(s_1',s_2')\right)^{1-\gamma}\pi(s_2')\pi(s_1'|s_1')\right)^{\frac{1}{\gamma}}$$

where the portfolio choices $(\theta_h^n(s_1),\theta_a^n(s_1))$ for each s_1 are the solution to

$$\begin{aligned}
&\max_{\theta_h,\theta_a}\sum_{s_1',s_2'}\left(\theta_h(1+r_h(s_1',s_2'))+\theta_a(s_1',s_2')\right)^{1-\gamma}\pi(s') \\
s.t.\quad &\theta_h+\sum_{s_1',s_2'}\frac{\theta_a(s_1',s_2')\pi(s_2')\pi(s_1'|s_1)}{1+r_f} \;=\; 1 \qquad (27)\\
&\theta_h(1+r_h(s_1',s_2'))+\theta_a(s_1',s_2') \;\geq\; \theta_h(1+r_h(s_1',s_2'))\left(\frac{\tilde{v}_d^n(s_1')}{\tilde{v}^n(s_1')}\right)^{\frac{1}{1-\gamma}}
\end{aligned}$$

The intensive-from value function and the corresponding optimal portfolio choice are obtained by taking the limit $\tilde{v}=\lim_{n\to\infty}\tilde{v}^n$, $\tilde{v}_d=\lim_{n\to\infty}\tilde{v}^n$, and $\theta=\lim_{n\to\infty}\theta^n$. To solve the portfolio problem (28) for given n and s_1, we first fix $\theta_h^n(s_1)=\bar{\theta}_h^n(s_1)$ and find $\theta_a^n(s_1)$ solving (28) for given $\bar{\theta}_h^n(s_1)$. To this end, order the states (s_1',s_2') for each s_1 so that $r_h(s_1,1)>r_h(s_1,2)>\ldots>r_h(s_1,S)$. Suppose for the first $(s_1',s_2')=1,\ldots,J(s_1)$ states the participation constraint is binding. Then $\theta_a^n(s_1)$ is given by

$$\begin{aligned}
\theta_a^n(s_1,s_1',s_2') \;&=\; \bar{\theta}_h^n(s_1)(1+r_h(s_1',s_2'))\left[\left(\frac{\tilde{v}_d^n(s_1')}{\tilde{v}^n(s_1')}\right)^{\frac{1}{1-\gamma}}-1\right] \quad for\;\;(s_1',s_2')=1,\ldots,J(s_1)\\
\theta_a^n(s_1,s_1',s_2') \;&=\; \bar{r}^n(s_1)\,-\bar{\theta}_h^n(s_1)(1+r_h(s_1',s_2'))\;\; for\;(s_1',s_2')=J(s_1)+1,\ldots,S \qquad (28)\\
\bar{r}^n(s_1) \;&=\; \frac{1}{\sum_{s'=J(s_1)+1}^{S}\pi(s_2')\pi(s_1'|s_1)}\Big[(1+r_f)(1-\bar{\theta}_h^n(s_1))\,+\\
&\quad\bar{\theta}_h^n(s_1)\sum_{s'=J(s_1)+1}^{S}(1+r_h(s_1',s_2'))\pi(s_2')\pi(s_1'|s_1)\,-\\
&\quad\bar{\theta}_h^n(s_1)\left(\left(\frac{\tilde{v}_d^n(s_1')}{\tilde{v}^n(s_1')}\right)^{\frac{1}{1-\gamma}}-1\right)\sum_{s'=1}^{J(s_1)}(1+r_h(s_1',s_2'))\pi(s_2')\pi(s_1'|s_1)\Big]
\end{aligned}$$

Deriving the corresponding first-order conditions it is easy to see that for given $\bar{\theta}_h^n(s_1)$, the solution $\theta_a^n(s_1)$ to (28) is determined by (29), where $J(s_1)$ is the smallest number for which the portfolio choice satisfies the participation constraint. The optimal $\theta_h^n(s_1)$ is then found using a grid and searching on the grid for the optimal value.

d) Calibration: Partial Equilibrium

In this section, we calibrate the partial equilibrium version of the economy, that is, we find values for the expected investment returns r_f and $\bar{r}(s_1) = \tilde{r}_h + \varphi(s_1) - \delta_h$ without specifying the production function that generates these returns. We calibrate an annual risk-free rate of $r_f = 3\%$, in line with Kaplan and Violante (2010) and roughly in line with Huggett et al. (2011) and Krueger and Perri (2006) who use a 4% annual risk-free rate, but also deduct capital income taxes.

We choose the age-dependent expected human capital returns, $\bar{r}(s_1)$, to match life-cycle profile of earnings of the median household in the data. Specifically, we first construct a life-cycle profile of median household earnings using data drawn from the Survey of Consumer Finance as in Krebs, Kuhn, and Wright (2015). Given this profile of median earnings, we compute the average earnings growth for the young households (age group $21-40$) and middle-aged household (age group $41-60$), and then choose $\bar{r}_h(y)$ and $\bar{r}_h(m)$ so that the model implies an earnings growth that match the empirical averages. The average annual earnings growth rate for young households is 4.10 percent and -0.76 percent for middle-aged households, and the implied excess human capital returns are $\bar{r}_h(y) - r_f = 6.77\%$ and $\bar{r}_h(m) - r_f = 1.65\%$. An average human capital return of 9.77 percent is in line with empirical estimates of the returns to on-the-job-training – see Krebs, Kuhn, and Wright (2005) for a discussion of the empirical literature. For old households we choose $\bar{r}_h(o) = 0$ so that these households only invest in physical capital (the risk-free asset).

We assume that human capital shocks, η, are approximately normally distributed, that is, we choose the probabilities $\pi(s_2)$ and the realizations $\eta(s_2)$ to approximate a normal distribution with mean 0 and standard deviation $\sigma_\eta = 0.15$. The parameter σ_η measures human capital risk and our choice of $\sigma_\eta = 0.15$ is motivated by the following considerations.

In the model economy, labor income of an individual household in period t is given by $y_{ht} = \tilde{r}_h h_t$, so that the growth rate of labor income is equal to the growth rate of human capital: $y_{h,t+1}/y_{ht} = h_{h,t+1}/h_t$. We can use the equilibrium solution to compute the human capital growth between year t and year $t+1$, which is

$$\frac{h_{t+1}}{h_t} \; = \; \beta \left(\theta_h(s_{1,t-1})(1 + \tilde{r}_h + \varphi(s_{1,t}) - \delta_h + \eta(s_{2t})) + \theta_a(s_{1,t-1}, s_{1t}, s_{2t}) \right) \tag{29}$$

If we neglect transitions across age groups s_1, then (29) can be written as

$$log y_{h,t+1} = log y_{ht} + d(s_1) + \epsilon_t, \tag{30}$$

where $d(s_1)$ is a constant and $\{\epsilon_t\}$ is a sequence of i.i.d. random variables with mean zero and variance

$$\sigma^2_\epsilon(s_1) \; = \; \theta^2_h(s_1) var \left[\theta_h(s_1)\eta(s_{2t}) + \theta_a(s_1, s_{2t}) \,|\, s_1 \right] \tag{31}$$

Hence, the logarithm of labor income follows a random walk with drift d and innovation term ϵ_t.[8] The random walk specification is often used by the empirical literature to model the permanent component of labor income risk (Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Storesletten et al. (2004)). Thus, their estimate of the standard deviation of the error term for the random walk component of annual labor income can be used to find a value for σ^2_η for given portfolio choices θ_h and θ_a. For young households, we will see below that θ_h is close to one and insurance payments, $\theta_a(s_2)$, are small, so that we have $\sigma^2_\epsilon \approx \sigma^2_\eta$. In our baseline calibration, we use $\sigma_\eta = .15$, which lies on the lower end of the spectrum of estimates found by the empirical literature. For example, Carroll and Samwick

[8] We have ϵ_t instead of ϵ_{t+1} in equation (30), and the latter is the common specification for a random walk. However, this is not a problem if the econometrician observes the idiosyncratic depreciation shocks with a one-period lag. In this case, (30) is the correct equation from the household's point of view, but a modified version of (30) with ϵ_{t+1} replacing ϵ_t is the specification estimated by the econometrician.

(1997) find .15, Meghir and Pistaferri (2004) estimate .19, and Storesletten et al. (2004) have .25 (averaged over age-groups and, if applicable, over business cycle conditions). All these studies use labor income before transfer payments, which is the relevant variable from our point of view.

For the baseline calibration, we assume that households who default regain access to financial markets after 7 years: $(1-p) = 1/7$. Finally, we assume a degree of relative risk aversion of $\gamma = 1$ (log-utility) and set the annual discount factor to $\beta = 0.95$.[9]

e) Consumption Insurance: Partial Equilibrium

In this section, we report the degree of consumption insurance that emerges in partial equilibrium. In addition, we discuss the model implication for human capital investment and the welfare consequences of the lack of consumption insurance. We measure human capital investment by the portfolio choice variable θ_h and consumption insurance by the variable $1 - \sigma_c/\sigma_{c,d}$, where σ_c is the standard deviation of individual consumption growth for the household with access to financial markets and $\sigma_{c,d}$ is the standard deviation of individual consumption growth for a household in default (no access to financial markets). Our welfare measure is the welfare gain, expressed as fraction of lifetime consumption, of moving from the current situation with imperfect insurance (binding enforcement constraint) to a situation with full insurance (no enforcement constraint). Our welfare measure expresses the cost of limited enforcement of financial contracts. The results are as follows.

For these parameter values, young households hold almost all of their wealth in human capital with $\theta_h = 0.98$. This seems reasonable given that households in their 20's typically have little in the way of net worth, while households in their thirties are approaching the peak of their human capital wealth. When households reach middle age, they hold a substantial

[9]An alternative calibration approach is to use the discount factor to match a given value of the expected human capital return for the young.

fraction of their wealth in the form of the risk free asset with the portfolio share of human capital falling to $\theta_h = 0.91$. The old, by assumption, hold no human capital.

Given that young households are almost entirely invested in their human capital, they are exposed to a large amount of risk. The enforcement constraints limit how much of this risk can be insured, with our measure of consumption insurance $1 - \sigma_c/\sigma_{c,d} = 0.434$ showing that almost 60% of the young's consumption risk is not insured. For middle aged households, who hold much more of the risk-free asset, consumption insurance rises to $1 - \sigma_c/\sigma_{c,d} = 0.758$ implying that middle-aged households are insured against all but about 25% of their consumption risk. Both of these numbers are significantly large than those derived from other studies in which there is no life-cycle profile for income and no investment in human capital; in those models, there is little reason for households to borrow so that enforcement constraints do not bind often and consumption insurance is high.

Lastly, the fact that consumption is underinsured leads to significant welfare losses. Young households would pay $\Delta = 3.503$ of their lifetime consumption to move to full insurance, while middle aged households would pay $\Delta = 1.418$ of their lifetime consumption. These numbers are large when compared to traditional measures of the costs of underinsurance, such as Lucas's (1987) welfare cost of business cycles.

f) Sensitivity Analysis: Partial Equilibrium

In this section, we analyze how the results of the partial equilibrium analysis depend on the various parameter choices. To this end, we consider a constant-type household; this means that our results will not be exactly comparable to those from the changing types model of the previous section.

Figure 1, 2, and 3 here.

We begin with a discussion how expected human capital returns affect portfolio choice, consumption insurance and welfare. Figure 1 depicts changes in the households human capital portfolio decision as excess returns vary. For excess returns on the order of 1.5%, which we found for middle aged households in the changing type economy, just over 85% of the households portfolio is invested in human capital. For returns approaching the 7% earned by young households, roughly 97% of the households portfolio is concentrated in human capital. As returns approach 15%, the portfolio share exceeds one implying that the household is borrowing to finance higher human capital investment.

Figure 2 plots consumption insurance as excess returns vary which further illustrates the steep drop off in consumption insurance as higher excess returns drive households onto their enforcement constraints. Whereas returns under 2% result in consumption insurance of 80% or higher, excess returns of around 7% drive insurance down under 40% falling to under 20% when excess returns exceed 14%. Figure 3 plots the corresponding welfare losses which rise from around 1% of lifetime consumption when excess returns are under 2% to more than 5% when excess returns exceed 6% all the way to 10% of lifetime consumption when excess returns are 16%.

Figures 4,5, and 6 here.

Figures 4, 5 and 6 depict what happens when we increase the coefficient of relative risk aversion from our benchmark $\gamma = 1$ to $\gamma = 2$. As shown in Figure 4, higher risk aversion leads to less investment in human capital at all levels of excess returns, with even a modest coefficient of 2 resulting in roughly five percentage points less investment in human capital.

As households shift away from human capital investment towards the risk free asset, the extent to which they are exposed to labor income risk is reduced. Figure 5 also shows that the share of this risk that can be insured increases. This is because, with higher risk aversion, the costs of being in financial autarky are larger resulting in looser enforcement constraints.

Figure 6 plots the corresponding welfare costs of imperfect enforcement. For low excess returns, welfare costs are similar for both levels of risk aversion. This is because the extra insurance available when risk aversion is higher is roughly offset by the lower consumption growth resulting from less investment in the high return asset. As excess returns grow, the human capital portfolio share levels out and so the higher consumption insurance leads to smaller welfare costs from underinsurance.

Figures 7, 8 and 9 here.

Figures 7, 8 and 9 depict the effect of varying the level of income risk from our benchmark of $\sigma_{eta} = 0.15$ to a lower value of $\sigma_{\eta} = 0.10$. With lower risk, Figure 7 shows that human capital investment levels are higher for most levels of the excess return. Only when excess returns exceed 14%, so that households begin to want to borrow to finance even greater human capital investments, does lower income risk lower human capital investment; this is because with lower levels of income risk, the cost of being in financial autarky is smaller and the enforcement constraints are tighter. Consumption insurance is always lower with lower income risk reflecting the tighter borrowing constraints as shown in Figure 8; for excess returns approaching 16%, consumption insurance approaches zero. However, as shown in Figure 9, the welfare costs of this underinsurance are smaller as the direct effect of reducing income risk dominates the reduction in the quantity of this risk that is insured.

Figure 10, 11, and 12 here.

Lastly, Figures 10, 11, and 12 depict the effect of changes in the severity of contract enforcement, with the amount of time the household is in financial autarky decreased from the benchmark average level of 7 years down to 4 years. Figure 10 shows that weaker enforcement leads to less investment in human capital, although the reduction is quite modest. Figures 11 and 12 show that both consumption insurance and the welfare costs of insurance are barely affected by changing the enforcement parameter.

g) Closing the Model: General Equilibrium

We now return to the full general equilibrium model (endogenous investment returns) and show how to complete the model calibration by introducing a production function and imposing a number of aggregate targets. We use a Cobb-Douglas production function $f(\tilde{K}) = A\tilde{K}^{\alpha}$, where $0 < \alpha < 1$ is capital's share in output and A is a productivity parameter. In this case, the rental rates of physical capital and human capital are given by

$$\begin{aligned} \tilde{r}_k &= \alpha A\tilde{K}^{\alpha-1} \\ \tilde{r}_h &= (1-\alpha)A\tilde{K}^{\alpha} \end{aligned} \tag{32}$$

As in Krebs, Kuhn, and Wright (2015), we target an aggregate share of capital income, $\tilde{r}_k K/Y$, of 0.32 so that $\alpha = 0.32$. We also follow Krebs, Kuhn, and Wright (2015) and target an aggregate capital-to-output ratio of 2.94. This target in conjunction with the target $r_f = \tilde{r}_k - \delta_k = 0.03$ yields $\tilde{r}_k = 0.1085$ and $\delta_k = 0.0785$.

The partial equilibrium model discussed in the previous sections implies an aggregate capital-to-output ratio that is given by the expression[10]

$$\frac{K}{Y} = \frac{1-\Theta_h}{\tilde{r}_k(1-\Theta_h) + \frac{\tilde{r}_h}{\phi}\Theta_h} \tag{33}$$

where $\Theta_h = \sum_{s_1} \theta_h(s_1)\Omega(s_1)$ is the aggregate share of human capital in total wealth. This aggregate portfolio choice is determined by the partial equilibrium model. Given the value $\tilde{r}_k = 0.1085$ and the target value $\frac{K}{Y} = 2.94$, equation (33) determines value of $\frac{\tilde{r}_h}{\phi}$. We find $\frac{\tilde{r}_h}{\phi} = 0.0772$.

Our final aggregate target is the ratio of human capital investment to output, $\frac{X_h}{Y}$. Using

[10] If the model matches a given capital-to-output ratio it also matches, for given $\tilde{r}_k$, a given capital income to output ratio, $\frac{\tilde{r}_k K}{Y}$.

the human capital accumulation equation and the equilibrium policies, we find

$$\frac{X_h}{Y} = \frac{\Theta_h \left(\sum_{s_1=y,m} g(s_1)\Omega(s_1) - \sum_{s_1=y,m} \bar{r}_h(s_1)\Omega(s_1) + \phi^2 \frac{\tilde{r}_h}{\phi} \right)}{\tilde{r}_k(1-\Theta_h) + \phi \frac{\tilde{r}_h}{\phi} \Theta_h} , \tag{34}$$

In (34) the values of all variables except ϕ are already determined. Thus, (34) can be used to pin down a value for ϕ once a target value for $\frac{X_h}{Y}$ is specified. We choose a target value $\frac{X_h}{Y} = 0.06$ – see Krebs, Kuhn, and Wright (2015) for a discussion of this choice. Given this target value, we find that $\phi = XX$. This together with the already specified value for $\frac{\tilde{r}_h}{\phi}$ implies $\tilde{r}_h = XX$. Further, the rental rate expressions (32) together with the values for $\tilde{r}_h$ and $\tilde{r}_k$ and α imply $A = XX$ and $\tilde{K} = XX$.

Finally, we note that the values for $\bar{r}_h(s_1)$ and $\tilde{r}_h$ imply values for the difference $\varphi(s_1) - \delta_h$, but that the learning-by-doing parameters and the human capital depreciation rate are not separately identified. However, if we follow Krebs, Kuhn, and Wright (2015) and choose $\delta_h = 0.04$, we find $\varphi(y) = XX$ and $\varphi(m) = XX$.

Appendix

Proof of Proposition 1

To simplify the notation, denote the endogenous individual state vector as $x_t = (w_t, \theta_t)$ and suppress the dependence on the aggregate state, Ω. Further, define a payoff function as $F(x_t, s_t, x_{t+1}) = u((1 + r(\theta_t, s_t))w_t - w_{t+1})$, and a feasibility correspondence as

$$\begin{aligned}\Gamma(x_t, s_t) \quad = \quad & \Big\{ x_{t+1} \in \mathbf{X} \mid \theta_{h,t+1}(s_t) + \sum_{s_{t+1}} \frac{\theta_{a,t+1}(s_{t+1})\pi(s_{t+1}|s_t)}{1 + r_f} = 1 \;, \\ & 0 \;\leq\; w_{t+1} \;\leq\; (1 + r(\theta_{t+1}, s_t))w_t \;\;,\;\; \theta_{h,t+1} \geq 0 \} \end{aligned} \qquad (A1)$$

Using this notation, the household maximization problem reads

$$\begin{aligned} max \quad & E\left[\sum_{t=0}^{\infty} \beta^t F(x_t, s_t, x_{t+1}) | x_0, s_0\right] \qquad (A2) \\ s.t. \quad & x_{t+1} \;\in\; \Gamma(x_t, s_t) \\ & E\left[\sum_{n=0}^{\infty} \beta^n F(x_{t+n}, s_{t+n}, x_{t+n+1}) | x_0, s^t\right] \;\geq\; V_d(x_t, s_t) \end{aligned}$$

The corresponding Bellman equation reads:

$$\begin{aligned} V(x, s) \quad = \quad & \max_{x'} \left\{ F(x, s, x') + \beta \sum_{s'} V(x', s')\pi(s'|s) \right\} \qquad (A3) \\ s.t. \quad & x' \;\in\; \Gamma(x, s) \\ & V(x', s') \;\geq\; V_d(x', s') \end{aligned}$$

Define an operator, T, that maps semi-continuous functions into semi-continuous functions as

$$\begin{aligned} TV(x, s) \quad = \quad & \max_{x'} \left\{ F(x, s, x') + \beta E[V(x', s')|s] \right\} \qquad (A4) \\ s.t. \quad & x' \in \Gamma(x, s) \\ & V(x's') \;\geq\; V_d(x', s') \;. \end{aligned}$$

A standard contraction mapping argument shows that there is a unique continuous solution, V_0, to the Bellman equation (A3) without participation constraint if i) F is continuous, ii) Γ is compact-valued and continuous, and (18) holds. Extending the argument of Rustichini (1998),[11] it can be shown that $V_\infty = \lim_{n\to\infty} T^n V_0$ exists, is equal to the maximal solution of the Bellman equation (A3), and is the value function of the sequential maximization problem (A2) if the following four conditions hold: i) F is continuous, ii) Γ is compact-valued and continuous, iii) for all states, (x, s), there exists a feasible plan for the sequential problem (A2) so that the corresponding expected lifetime utility (payoff) is greater than $-\infty$, and iv) for any given state, (x, s), the value function of the max-problem without participation constraints satisfies $V_0^*(x, s) < +\infty$. Thus, to prove proposition 1 it suffices to show that conditions i)-iv) hold.

The continuity of the payoff function, F, is obvious. The correspondence, Γ, is compact-valued since portfolio-choices, θ', are elements of a closed and bounded subset of $\mathbb{R}^m$. Closedness follows from the fact that the set is defined by equalities and weak inequalities. Restricting attention to a bounded set can be shown to be without loss of generality. Continuity of the correspondence Γ is also straightforward to show. A standard argument shows that conditions iii) and iv) hold if condition (18) is satisfied. This proves proposition 2.

Proof of Proposition 2

As before, let V_0 be the solution of the Bellman equation (A3) without the participation constraint. Simple guess-and-verify shows that V_0 has the following functional form:

$$V_0(w, \theta, s) \quad = \quad \begin{cases} \tilde{V}_0(s)\,(1 + r(\theta, s))^{1-\gamma}\, w^{1-\gamma} & if\ \gamma\ \neq\ 1 \\ \tilde{V}_0(s)\ +\ \frac{1}{1-\beta} log(1 + r(\theta, s))\ +\ \frac{1}{1-\beta} w & otherwise \end{cases} \qquad (A5)$$

[11]Rustichini (1998) consider a class of dynamic programming problems with participation constraint (incentive compatability constraint) and possibly unbounded utility. However, he requires bi-convergence, which is always satisfied if lifetime-utility is bounded for all feasible paths (Streufert, 1990). Unfortunately, in our problem with $\gamma \geq 1$ the requirement of lower convergence is not satisfied, so that Rustichini (1998) is not directly applicable.

where $\tilde{V}_0$ is the solution to the intensive-form Bellman equation (A3) without participation constraint. Let the operator T be defined as in (A4). We show by induction that if $V_n = T^n V_0$ has the functional form, then $V_{n+1} = T^{n+1} V_)$ has the functional form. For $n = 0$ the claim is true because V_0 has the functional form. Suppose now V_n has the functional form. We then have

$$
\begin{aligned}
V_{n+1}(w,\theta,s) &= TV_n(w,\theta,s) \\
&= \max_{w',\theta'} \left\{ \frac{((1+r(\theta,s))w - w')^{1-\gamma}}{1-\gamma} + \sum_{s'} \tilde{V}_n(s')(1+r(\theta',s'))^{1-\gamma}(w')^{1-\gamma}\pi(s'|s) \right\} \\
s.t.\quad 1 &= \theta'_h + \sum_{s'} \frac{\theta'_a(s')\pi(s'|s)}{1+r_f} \qquad\qquad (A6) \\
& 0 \le w' \le (1+r(\theta,s))w \ , \ \theta'_h \ge 0 \\
& \tilde{V}_n(s')\left(1+r(\theta'_h,\theta_a(s'))\right)^{1-\gamma}(w')^{1-\gamma} \\
& \ge \tilde{V}_d(s')(1+r(\theta'_h,0,s')
\end{aligned}
$$

for $\gamma \neq 1$ and

$$
\begin{aligned}
V_{n+1}(w,\theta,s) &= TV_n(w,\theta,s) \\
&= \max_{w',\theta'} \Bigg\{ log\,(1+r(\theta,s))w - w') + \beta \sum_{s'} \tilde{V}_n(s')\pi(s'|s) \\
&\qquad + \frac{\beta}{1-\beta}\sum_{s'} log(1+r(\theta',s'))\pi(s'|s) + \frac{\beta}{1-\beta} log\, w' \Bigg\} \\
s.t.\quad 1 &= \theta'_h + \sum_{s'} \frac{\theta'_a(s')\pi(s'|s)}{1+r_f} \\
& 0 \le w' \le (1+r(\theta,s))w \ , \ \theta'_h \ge 0 \\
& \tilde{V}_n(s') + \frac{1}{1-\beta} log(1+r(\theta'_h,\theta'_a(s'),s')) + \frac{1}{1-\beta} log\, w' \\
& \ge \tilde{V}_d(s') + \frac{1}{1-\beta} log(1+r(\theta'_h,0,s')) + \frac{1}{1-\beta} log\, w'
\end{aligned}
$$

for the log-utility case. Clearly, the solution to the maximization problem defined by the right-hand-side of (A6) has the form

$$
w'_{n+1} = (1-\tilde{c}_{n+1}(s))(1+r(\theta_{n+1},s))w \qquad\qquad (A7)
$$

$$\theta'_{n+1} \quad = \quad \theta'_{n+1}(s) \ ,$$

where the subscript $n+1$ indicates step $n+1$ in the iteration. Thus, we have

$$V_{n+1}(w, \theta, s) \quad = \quad \begin{cases} \tilde{V}_{n+1}(s) \left(1 + r(\theta, s)\right)^{1-\gamma} w^{1-\gamma} & if \ \gamma \neq 1 \\ \tilde{V}_{n+1}(s) \ + \ \frac{1}{1-\beta} log(1 + r(\theta, s)) \ + \ \frac{1}{1-\beta} w & otherwise \end{cases} \ ,$$

where $\tilde{V}_{n+1}$ is defined accordingly.

From proposition 2 we know that $V_\infty = \lim_{n\to\infty} T^n V_0$ exists and that it is the maximal solution to the Bellman equation (A3) as well as the value function of the corresponding sequential maximization problem (A2). Since the set of functions with this functional form is a closed subset of the set of semi-continuous functions, we know that V_∞ has the functional form. This prove proposition 3.

Proof of Proposition 3

From proposition 2 we know that individual households maximize utility subject to the budget constraint and participation constraint if condition (18) is satisfied. Thus, it remains to show that the market clearing condition can be written as (16) and that the law of motion (22) describes the equilibrium evolution of the relative wealth distribution.

For the aggregate value of financial asset holdings we find:

$$\begin{aligned} E[\theta_{a,t+1} w_{t+1}] &= E[\theta_{a,t+1}(1 - \tilde{c}_t)(1 + r_t) w_t] \qquad (A8) \\ &= E[E[\theta_{a,t+1}(1 - \tilde{c}_t)(1 + r_t) w_t | s_t]] \\ &= E[\theta_{a,t+1}(1 - \tilde{c}_t) E[(1 + r_t) w_t | s_t]] \\ &= E\left[(1 + r_t) w_t\right] \ \frac{E[\theta_{a,t+1}(1 - \tilde{c}_t) E[(1 + r_t) w_t | s_t]]}{E\left[(1 + r_t) w_t\right]} \\ &= E\left[(1 + r_t) w_t\right] \ E\left[\theta_{a,t+1}(1 - \tilde{c}_t) \Omega(s_t)\right] \ . \end{aligned}$$

where the first line follows from the budget constraint, the second line from the law of iterated expectations, the third line from the fact that $\theta_{a,t+1}$ and $\tilde{c}_t$ are independent of wealth and

s^{t-1}, and the last line from the definition of Ω. A similar argument shows that

$$H_{t+1} = E\left[(1+r_t)w_t\right] E\left[\theta_{h,t+1}(1-\tilde{c}_t)\Omega(s_t)\right]. \qquad (A9)$$

Using the market clearing condition (12) and the definition of $\tilde{K}$ shows that (22) has to hold.

Finally, the law of motion for Ω can be found as:

$$\begin{aligned}
\Omega_{t+1}(s_{t+1}) &= \frac{E\left[(1+r_{t+1})w_{t+1}|s_{t+1}\right]}{E\left[(1+r_{t+1})w_{t+1}\right]} \qquad (A10) \\
&= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)(1+r_t)w_t|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)(1+r_t)w_t\right]} \\
&= \frac{E\left[E\left[(1+r_{t+1})(1-\tilde{c}_t)(1+r_t)w_t|s_t\right]|s_{t+1}\right]}{E\left[E\left[(1+r_{t+1})(1-\tilde{c}_t)(1+r_t)w_t|s_t\right]\right]} \\
&= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[(1+r_t)w_t|s_t\right]|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[(1+r_t)w_t|s_t\right]\right]} \\
&= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[(1+r_t)w_t|s_t\right]|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)E\left[(1+r_t)w_t|s_t\right]\right]} \times \\
&\quad \frac{E\left[(1+r_t)w_t\right]}{E\left[(1+r_t)w_t\right]} \\
&= \frac{E\left[(1+r_{t+1})(1-\tilde{c}_t)\Omega_t(s_t)|s_{t+1}\right]}{E\left[(1+r_{t+1})(1-\tilde{c}_t)\Omega_t(s_t)\right]},
\end{aligned}$$

where the second line follows from the budget constraint, the third line from the law of iterated expectations, the fourth line from the fact that θ_{t+1} and $\tilde{c}_t$ are independent of wealth and s^{t-1}, and the last line from the definition of Ω. This completes the proof of proposition 3.

References

Alvarez, F., and U. Jermann (2000) "Efficiency, Equilibrium, and Asset Pricing With Risk of Default," *Econometrica* 68: 775-797.

Andolfatto, D., and M. Gervais (2006) "Human Capital Investment and Debt Constraints," *Review of Economic Dynamics* 10: 1-30.

Ben-Porath, Y. "The Production of Human capital and the Life-Cycle of Earnings," *Journal of Political Economy* 75: 352-365.

Blundell, R., Pistaferri, L., and I. Peston (2008) "Consumption Inequality and Partial Insurance," *American Economic Review* forthcoming.

Carroll, C., and A. Samwick (1997) "The Nature of Precautionary Wealth," *Journal of Monetary Economics* 40: 41-72.

Cordoba, J., (2008) "US Inequality: Debt Constraints of Incomplete Markets?," *Journal of Monetary Economics* 55: 350-364.

Geweke, J., and M. Keane (2000) "An Empirical Analysis of Male Income Dynamics in the PSID: 1968-1989," *Journal of Econometrics* 96: 293-356.

Guvenen, F., Kuruscu, B., and S. Ozkan (2011) "Taxation of Human Capital and Wage Inequality," Working Paper.

Hellwig, C., and G. Lorenzoni (2009) "Bubbles and Self-Enforcing Debt," *Econometrica* 77: 1137-1164.

Huggett, M., Ventura, G., and A. Yaron (2011) "Source of Lifetime Inequality," *American Economic Review* 101: 2923-54.

Jacobson, L., LaLonde, R., and D. Sullivan (1993) "Earnings Losses of Displaced Workers," *American Economic Review* 83: 685-709.

Jones, L., and R. Manuelli (1990) "A Convex Model of Equilibrium Growth: Theory and Policy Implications," *Journal of Political Economy* 98: 1008-1038.

Kehoe, T., and D. Levine (1993) "Debt-Constrained Asset Markets," *Review of Economic Studies* 60: 868-888.

Kocherlakota, N. (1996) "Implications of Efficient Risk Sharing Without Commitment," *Review of Economic Studies* 63: 595-609.

Krebs, T. (2003) “Human Capital Risk and Economic Growth,” *The Quarterly Journal of Economics* 118: 709-744.

Krebs, T., Kuhn, M., and M. Wright (2015) “Human Capital Risk, Contract Enforcement, and the Macroeconomy,” *American Econmic Review* forthcoming.

Krueger, D., and F. Perri (2006) “Does Income Inequality Lead to Consumption Inequality? Evidence and Theory” *Review of Economics Studies*, 73: 163-193.

Krueger, D., and H. Uhlig (2006) “Competitive Risk Sharing Contracts with One-Sided Limited Commitment,” *Journal of Monetary Economics* 53: 1661-1691.

Ligon, E., J. Thomas, and T. Worrall (2002) “Informal Insurance: Arrangements with Limited Commitment: Theory and Evidence from Village Economies,” *Review of Economic Studies* 69: 209-244.

Lochner, L., and A. Monge-Naranjo (2011) “The Nature of Credit Constraints and Human Capital,” *American Economic Review* 101: 2487-2529.

MaCurdy, T. (1982) “The Use of Time Series Processes to Model the Error Structure of Earnings in a Longitudinal Data Analysis,” *Journal of Econometrics* 18: 83-114.

Meghir, C. and L. Pistaferri (2004) “Income Variance Dynamics and Heterogeneity,” *Econometrica* 72: 1-32.

Rustichini, A. (1998) “Dynamic Programming Solution of Incentive Constrained Problems,” *Journal of Economic Theory* 78: 329-354.

Storesletten, K., Telmer, C., and A. Yaron (2004) “Cyclical Dynamics of Labor Market Risk,” *Journal of Political Economy*

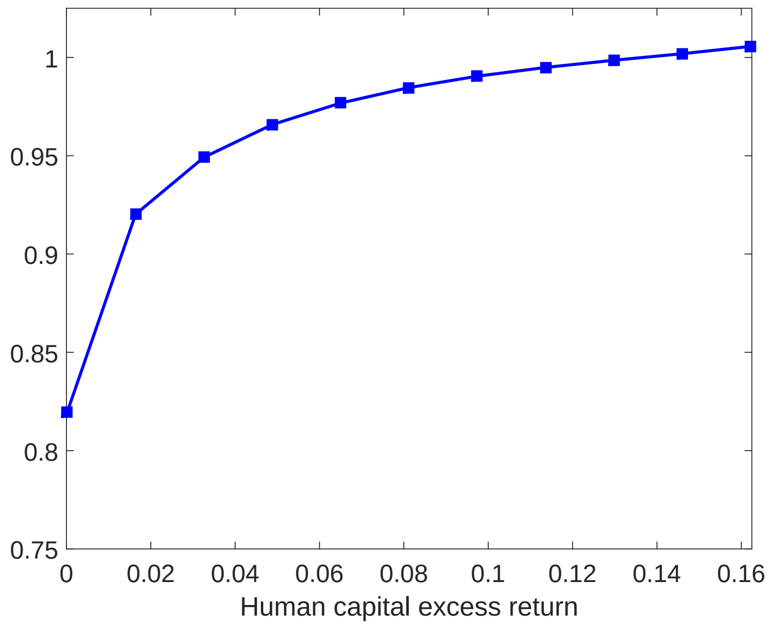
Streufert, P. (1990) “Stationary Recursive Utility and Dynamic Programming under the Assumption of Biconvergence,” *Review of Economic Studies* 57: 79-97.

Thomas, J., and T. Worrall (1988) “Self-Enforcing Wage Contracts,” *Review of Economic Studies* 55: 541-554.

Trostel, P. (1993) “The Effect of Taxation on Human Capital,” *Journal of Political Economy* 101: 327-350.

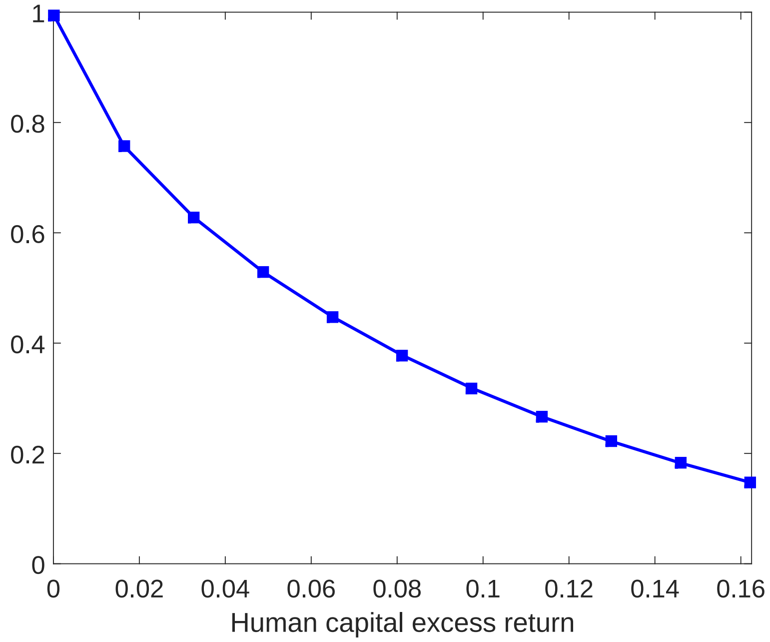
Wright, M. (2001) “Investment and Growth with Limited Commitment,” Working Paper, Stanford University.

Figure 1: Portfolio choice for benchmark model



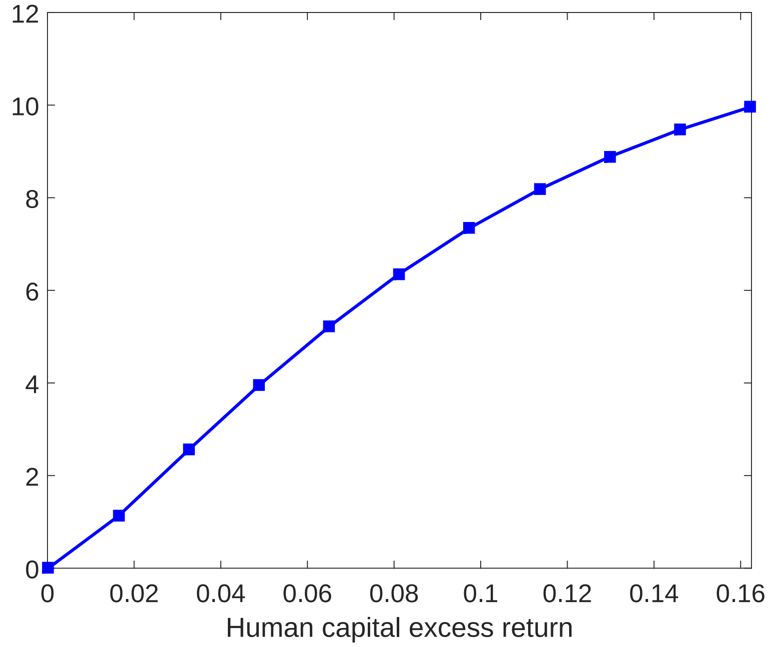
Notes: Human capital share in total wealth as a function of excess human capital returns in the baseline model.

Figure 2: Consumption insurance for benchmark model



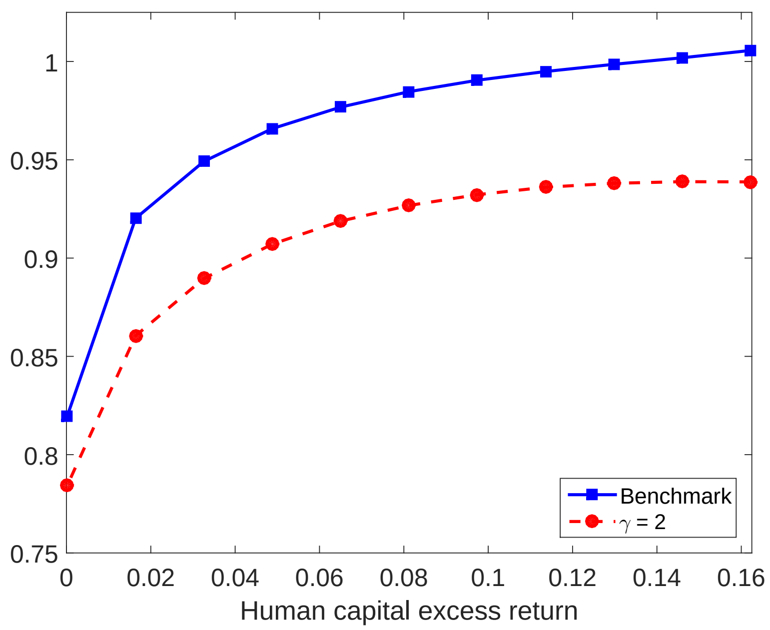
Notes: Consumption insurance as a function of excess human capital returns in the baseline model. The insurance measure is one minus the ratio of the standard deviation of consumption relative to the standard deviation of consumption in financial autarky.

Figure 3: Welfare cost of underinsurance for benchmark model



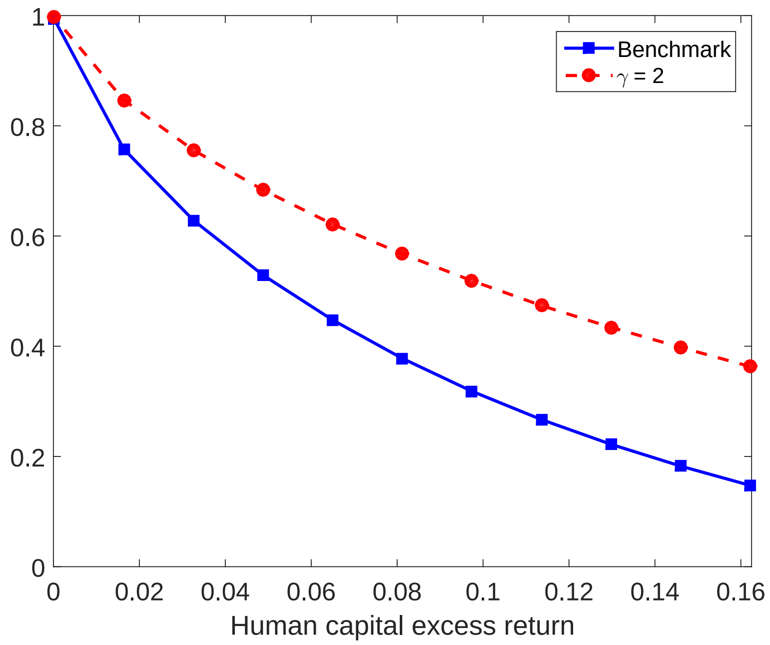
Notes: Welfare cost of underinsurance as a function of excess human capital returns in the baseline model. Welfare costs are the equivalent variation in lifetime consumption for a household without constraints. Welfare costs are in percentage points.

Figure 4: Portfolio choice for different degrees of risk aversion



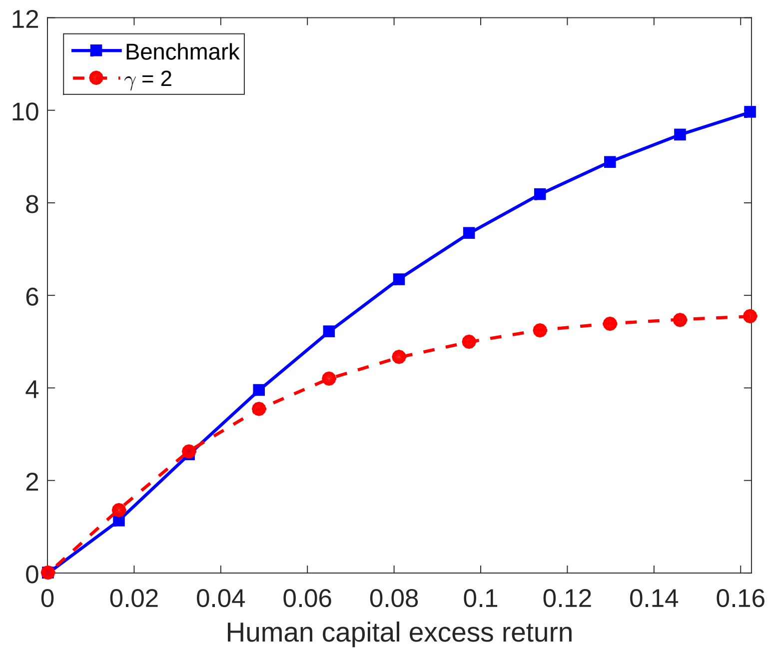
Notes: Human capital share in total wealth as a function of excess human capital returns in the baseline model ($\gamma = 1$, log utility) in comparison to a case with higher risk aversion ($\gamma = 2$).

Figure 5: Consumption insurance for different degrees of risk aversion



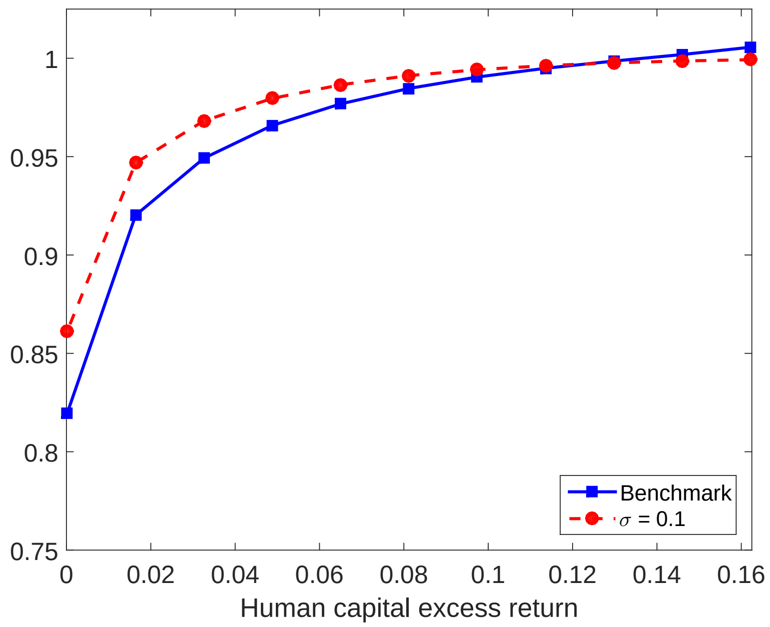
Notes: Consumption insurance as a function of excess human capital returns in the baseline model ($\gamma = 1$, log utility) in comparison to a case with higher risk aversion ($\gamma = 2$). The insurance measure is one minus the ratio of the standard deviation of consumption relative to the standard deviation of consumption in financial autarky.

Figure 6: Welfare cost of underinsurance for different degrees of risk aversion



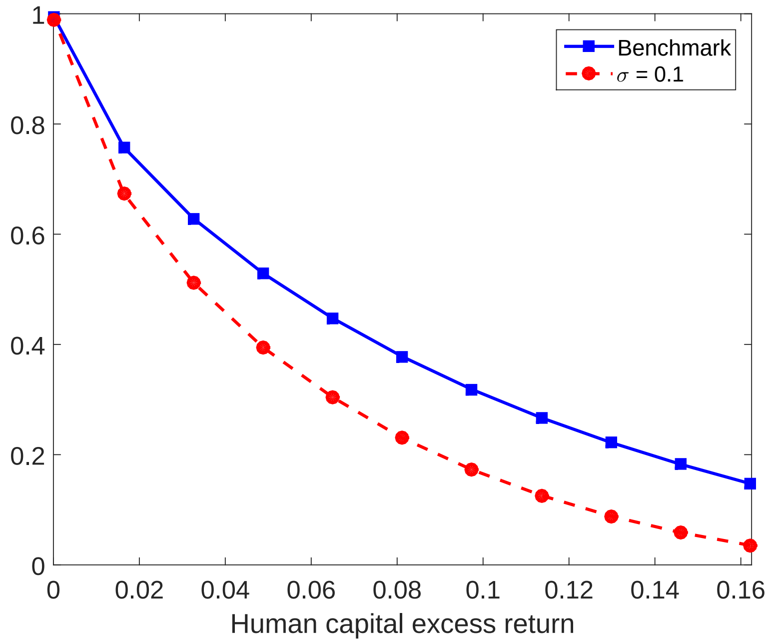
Notes: Welfare cost of underinsurance as a function of excess human capital returns in the baseline model ($\gamma = 1$, log utility) in comparison to a case with higher risk aversion ($\gamma = 2$). Welfare costs are the equivalent variation in lifetime consumption for a household without constraints. Welfare costs are in percentage points.

Figure 7: Portfolio choice for different levels of income risk



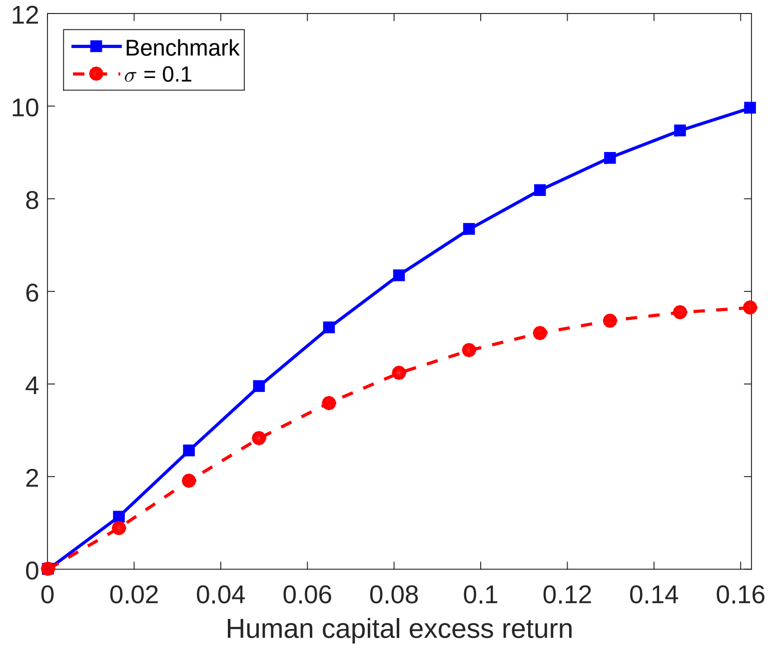
Notes: Human capital share in total wealth as a function of excess human capital returns in the baseline model ($\sigma = 0.15$) in comparison to a case with lower income risk ($\sigma = 0.1$).

Figure 8: Consumption insurance for different levels of income risk



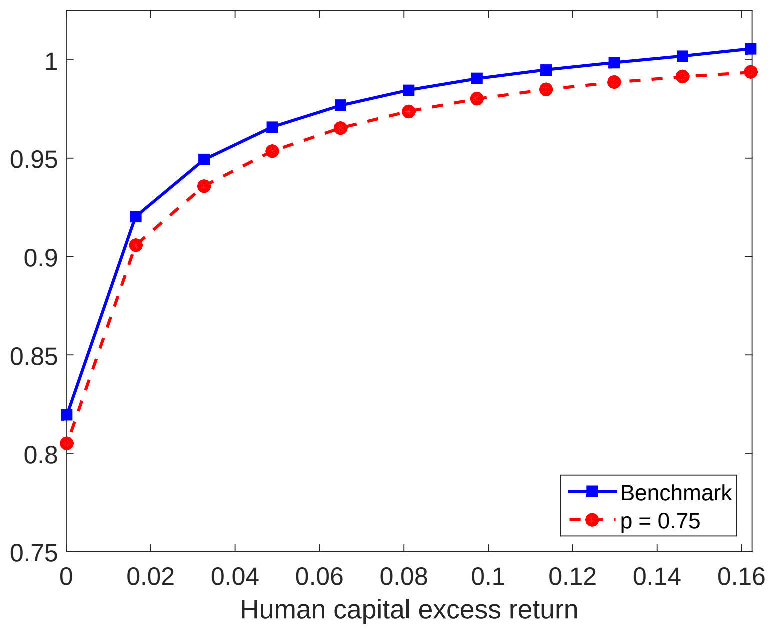
Notes: Consumption insurance as a function of excess human capital returns in the baseline model ($\sigma = 0.15$) in comparison to a case with lower income risk ($\sigma = 0.1$). The insurance measure is one minus the ratio of the standard deviation of consumption relative to the standard deviation of consumption in financial autarky.

Figure 9: Welfare cost of underinsurance for different levels of income risk



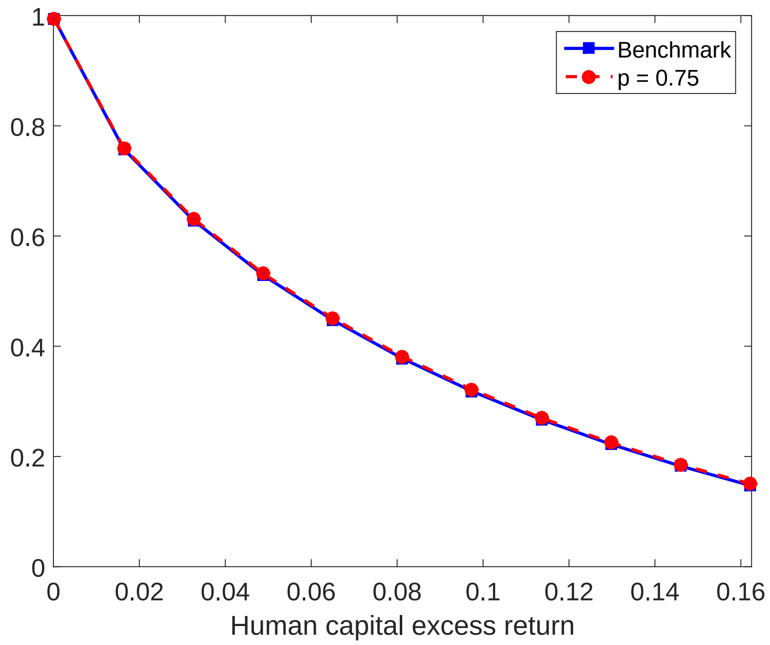
Notes: Welfare cost of underinsurance as a function of excess human capital returns in the baseline model ($\sigma = 0.15$) in comparison to a case with lower income risk ($\sigma = 0.1$). Welfare costs are the equivalent variation in lifetime consumption for a household without constraints. Welfare costs are in percentage points.

Figure 10: Portfolio choice for different levels of enforcement



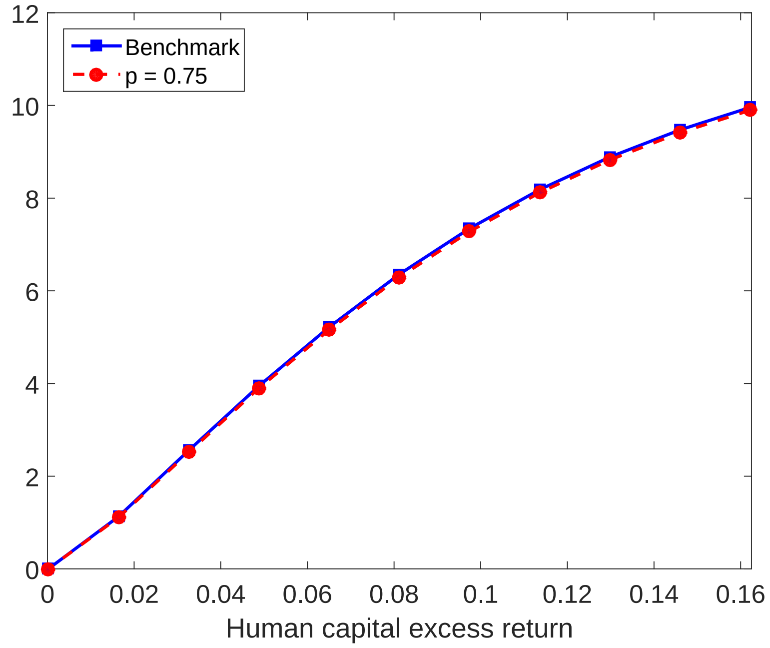
Notes: Human capital share in total wealth as a function of excess human capital returns in the baseline model ($\sigma = 0.15$) in comparison to a case with lower income risk ($\sigma = 0.1$).

Figure 11: Consumption insurance for different levels of enforcement



Notes: Consumption insurance as a function of excess human capital returns in the baseline model ($p = \frac{6}{7}$) in comparison to a case with weaker enforcement ($p = \frac{3}{4}$). The insurance measure is one minus the ratio of the standard deviation of consumption relative to the standard deviation of consumption in financial autarky.

Figure 12: Welfare cost of underinsurance for different levels of enforcement



Notes: Welfare cost of underinsurance as a function of excess human capital returns in the baseline model ($p = \frac{6}{7}$) in comparison to a case with weaker enforcement ($p = \frac{3}{4}$). Welfare costs are the equivalent variation in lifetime consumption for a household without constraints. Welfare costs are in percentage points.